Generalized Bianchi identities in gauge-natural field theories and the curvature of variational principles*

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Abstract

By resorting to Noether's Second Theorem, we relate the generalized Bianchi identities for Lagrangian field theories on gauge-natural bundles with the kernel of the associated gauge-natural Jacobi morphism. A suitable definition of the curvature of gauge-natural variational principles can be consequently formulated in terms of the Hamiltonian connection canonically associated with a generalized Lagrangian obtained by contracting field equations.

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1 Introduction

To investigate conservation laws for covariant field theories, see *e.g.* [1] and references quoted therein, the general problem has been tackled of coherently defining the lifting of infinitesimal transformations of the basis manifolds to the bundle of fields (namely bundles of tensor fields or tensor densities which could be obtained as suitable representations of the action of infinitesimal space-time transformations on frame bundles of a given order [23]). Such theories were also called geometric or *natural* [28]. An important generalization of natural theories to gauge fields theories passed through the concept of jet prolongation of a principal bundle and the introduction of a very important geometric construction, namely the *gauge-natural* bundle functor [4, 17].

Within the above mentioned general program generalized Bianchi identities for geometric field theories were introduced to get (after an integration by parts procedure) a consistent equation involving divergences within the first variation formula. It was also stressed that in the general theory of relativity these identities coincide with the contracted Bianchi identities for the curvature tensor of the pseudo-Riemannian metric.

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Our general framework is the calculus of variations on finite order fibered bundles. Fibered bundles will be assumed to be *gauge-natural bundles* (*i.e.* jet prolongations of fiber bundles associated to some gauge-natural prolongation of a principal bundle $\boldsymbol{P}$ [4, 17]). Such geometric structures have been widely recognized to suitably describe so-called gauge-natural field theories, *i.e.* physical theories in which right-invariant infinitesimal automorphisms of the structure bundle $\boldsymbol{P}$ uniquely define the transformation laws of the fields themselves (see *e.g.* [5] and references quoted therein). We shall in particular consider *finite order variational sequences on gauge-natural bundles*. The gauge-natural lift enables one to define the generalized gauge-natural Jacobi morphism where the *variation vector fields* are Lie derivatives of sections of the gauge-natural bundle with respect to gauge-natural lifts of infinitesimal automorphisms of the underlying principal bundle $\boldsymbol{P}$. Within such a picture, as a consequence of the Second Noether Theorem, it is possible to relate the generalized Bianchi morphism to the second variation of the Lagrangian and then to the associated Jacobi morphism [8, 9, 25, 26]. In the case of geodesics in a Riemannian manifold vector fields which make the second variation to vanish identically modulo boundary terms are called *Jacobi fields* and they are solutions of a second–order differential equation known as *Jacobi equation* (of geodesics). The notion of Jacobi equation as an outcome of the second variation is in fact fairly more general: formulae for the second variation of a Lagrangian functional and generalized Jacobi equations along critical sections have been already considered (see, *e.g.* the results of [2] and classical refrences quoted therein). In this paper we show that, as a consequence of the gauge-natural invariance of the Lagrangian, there exists a covariantly conserved current associated with the contraction of the Euler–Lagrange morphism with a gauge-natural Jacobi vector field. Such a conserved current can be considered as a Hamiltonian for the Lagrangian corresponding to such a contraction. The curvature of the variational principle can be then defined as the curvature of the corresponding Hamiltonian connection.

2 Finite order jets of gauge-natural bundles and variational sequences

In this Section we recall some basic facts about jet spaces. We introduce jet spaces of a fibered manifold and the sheaves of forms on the s–th order jet space. Moreover, we recall the notion of horizontal and vertical differential [17, 27].

Our framework is a fibered manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$.

For $s \geq q \geq 0$ integers we are concerned with the s–jet space $J_s\boldsymbol{Y}$ of s–jet prolongations of (local) sections of π; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fiberings $\pi^s_q : J_s\boldsymbol{Y} \to J_q\boldsymbol{Y}$, $s \geq q$, $\pi^s : J_s\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the *affine* fiberings π^s_{s-1}. We denote by $V\boldsymbol{Y}$ the vector subbundle of the tangent bundle $T\boldsymbol{Y}$ of vectors on $\boldsymbol{Y}$ which are vertical with respect to the fibering π.

Greek indices $\sigma, \mu, \dots$ run from 1 to n and they label basis coordinates,

while Latin indices $i, j, \ldots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote multi–indices of dimension n by boldface Greek letters such as $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, with $0 \leq \alpha_\mu$, $\mu = 1, \ldots, n$; by an abuse of notation, we denote by σ the multi–index such that $\alpha_\mu = 0$, if $\mu \neq \sigma$, $\alpha_\mu = 1$, if $\mu = \sigma$. We also set $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$ and $\boldsymbol{\alpha}! := \alpha_1! \ldots \alpha_n!$. The charts induced on $J_s\boldsymbol{Y}$ are denoted by $(x^\sigma, y^i_{\boldsymbol{\alpha}})$, with $0 \leq |\boldsymbol{\alpha}| \leq s$; in particular, we set $y^i_{\boldsymbol{0}} \equiv y^i$. The local vector fields and forms of $J_s\boldsymbol{Y}$ induced by the above coordinates are denoted by $(\partial^{\boldsymbol{\alpha}}_i)$ and $(d^i_{\boldsymbol{\alpha}})$, respectively.

For $s \geq 1$, we consider the natural complementary fibered morphisms over $J_s\boldsymbol{Y} \to J_{s-1}\boldsymbol{Y}$ (see [19, 20, 29]):

$$\mathcal{D} : J_s\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{s-1}\boldsymbol{Y}\,, \qquad \vartheta : J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} TJ_{s-1}\boldsymbol{Y} \to VJ_{s-1}\boldsymbol{Y}\,,$$

with coordinate expressions, for $0 \leq |\boldsymbol{\alpha}| \leq s-1$, given by

$$\mathcal{D} \quad = d^\lambda \otimes \mathcal{D}_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\boldsymbol{\alpha}+\lambda} \partial^{\boldsymbol{\alpha}}_j)\,, \vartheta \quad = \vartheta^j_{\boldsymbol{\alpha}} \otimes \partial^{\boldsymbol{\alpha}}_j = (d^j_{\boldsymbol{\alpha}} - y^j_{\boldsymbol{\alpha}+\lambda} d^\lambda) \otimes \partial^{\boldsymbol{\alpha}}_j\,.$$

The morphisms above induce the following natural splitting (and its dual):

$$J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} T^*J_{s-1}\boldsymbol{Y} = \left(J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X} \right) \oplus \mathcal{C}^*_{s-1}[\boldsymbol{Y}]\,, \tag{1}$$

where $\mathcal{C}^*_{s-1}[\boldsymbol{Y}] := \operatorname{im} \vartheta^*_s$ and $\vartheta^*_s : J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} V^*J_{s-1}\boldsymbol{Y} \to J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} T^*J_{s-1}\boldsymbol{Y}$.

If $f : J_s\boldsymbol{Y} \to I\!R$ is a function, then we set $D_\sigma f := \mathcal{D}_\sigma f$, $D_{\boldsymbol{\alpha}+\sigma} f := D_\sigma D_{\boldsymbol{\alpha}} f$, where D_σ is the standard *formal derivative*. Given a vector field $\Xi : J_s\boldsymbol{Y} \to TJ_s\boldsymbol{Y}$, the splitting (1) yields $\Xi \circ \pi^{s+1}_s = \Xi_H + \Xi_V$ where, if $\Xi = \Xi^\gamma \partial_\gamma + \Xi^i_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}_i$, then we have $\Xi_H = \Xi^\gamma D_\gamma$ and $\Xi_V = (\Xi^i_{\boldsymbol{\alpha}} - y^i_{\boldsymbol{\alpha}+\gamma} \Xi^\gamma) \partial^{\boldsymbol{\alpha}}_i$. We shall call Ξ_H and Ξ_V the horizontal and the vertical part of Ξ, respectively.

The splitting (1) induces also a decomposition of the exterior differential on $\boldsymbol{Y}$, $(\pi^s_{s-1})^* \circ d = d_H + d_V$, where d_H and d_V are defined to be the *horizontal* and *vertical differential*. The action of d_H and d_V on functions and 1–forms on $J_s\boldsymbol{Y}$ uniquely characterizes d_H and d_V (see, *e.g.*, [27, 29] for more details). A *projectable vector field* on $\boldsymbol{Y}$ is defined to be a pair (u, ξ), where $u : \boldsymbol{Y} \to T\boldsymbol{Y}$ and $\xi : \boldsymbol{X} \to T\boldsymbol{X}$ are vector fields and u is a fibered morphism over ξ. If there is no danger of confusion, we will denote simply by u a projectable vector field (u, ξ). A projectable vector field (u, ξ) can be conveniently prolonged to a projectable vector field $(j_s u, \xi)$; coordinate expression can be found *e.g.* in [19, 27, 29].

2.1 Gauge-natural bundles

Let $\boldsymbol{P} \to \boldsymbol{X}$ be a principal bundle with structure group $\boldsymbol{G}$. Let $r \leq k$ be integers and $\boldsymbol{W}^{(r,k)}\boldsymbol{P} := J_r\boldsymbol{P} \underset{\boldsymbol{X}}{\times} L_k(\boldsymbol{X})$, where $L_k(\boldsymbol{X})$ is the bundle of k–frames in $\boldsymbol{X}$ [4, 17], $\boldsymbol{W}^{(r,k)}\boldsymbol{G} := J_r\boldsymbol{G} \odot GL_k(n)$ the semidirect product with respect to the action of $GL_k(n)$ on $J_r\boldsymbol{G}$ given by the jet composition and $GL_k(n)$ is the group of k–frames in $I\!R^n$. Here we denote by $J_r\boldsymbol{G}$ the space of (r, n)-velocities on $\boldsymbol{G}$.

The bundle $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ is a principal bundle over $\boldsymbol{X}$ with structure group $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$. Let $\boldsymbol{F}$ be any manifold and $\zeta : \boldsymbol{W}^{(r,k)}\boldsymbol{G} \times \boldsymbol{F} \to \boldsymbol{F}$ be a left action of $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ on $\boldsymbol{F}$. There is a naturally defined right action of $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ on $\boldsymbol{W}^{(r,k)}\boldsymbol{P} \times \boldsymbol{F}$ so that we can associate in a standard way to $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ the bundle, on the given basis $\boldsymbol{X}$, $\boldsymbol{Y}_\zeta := \boldsymbol{W}^{(r,k)}\boldsymbol{P} \times_\zeta \boldsymbol{F}$.

Definition 1 We say $(\boldsymbol{Y}_\zeta, \boldsymbol{X}, \pi_\zeta; \boldsymbol{F}, \boldsymbol{G})$ to be the *gauge-natural bundle* of order (r,k) associated to the principal bundle $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ by means of the left action ζ of the group $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ on the manifold $\boldsymbol{F}$ [4, 17]. □

Remark 1 A principal automorphism Φ of $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ induces an automorphism of the gauge-natural bundle by:

$$\Phi_\zeta : \boldsymbol{Y}_\zeta \to \boldsymbol{Y}_\zeta : [(j^x_r\gamma, j^0_k t), \hat{f}]_\zeta \mapsto [\Phi(j^x_r\gamma, j^0_k t), \hat{f}]_\zeta \,, \tag{2}$$

where $\hat{f} \in \boldsymbol{F}$ and $[\cdot,\cdot]_\zeta$ is the equivalence class induced by the action ζ. □

Definition 2 We define the *vector* bundle over $\boldsymbol{X}$ of right–invariant infinitesimal automorphisms of $\boldsymbol{P}$ by setting $\mathcal{A} = T\boldsymbol{P}/\boldsymbol{G}$.

We also define the *vector* bundle over $\boldsymbol{X}$ of right invariant infinitesimal automorphisms of $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ by setting $\mathcal{A}^{(r,k)} := T\boldsymbol{W}^{(r,k)}\boldsymbol{P}/\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ $(r \leq k)$. □

Denote by $\mathcal{T}_{\boldsymbol{X}}$ and $\mathcal{A}^{(r,k)}$ the sheaf of vector fields on $\boldsymbol{X}$ and the sheaf of right invariant vector fields on $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$, respectively. A functorial mapping $\mathfrak{G}$ is defined which lifts any right–invariant local automorphism (Φ, ϕ) of the principal bundle $W^{(r,k)}\boldsymbol{P}$ into a unique local automorphism (Φ_ζ, ϕ) of the associated bundle $\boldsymbol{Y}_\zeta$. Its infinitesimal version associates to any $\bar{\Xi} \in \mathcal{A}^{(r,k)}$, projectable over $\xi \in \mathcal{T}_{\boldsymbol{X}}$, a unique *projectable* vector field $\hat{\Xi} := \mathfrak{G}(\bar{\Xi})$ (the gauge-natural lift) on $\boldsymbol{Y}_\zeta$ in the following way:

$$\mathfrak{G} : \boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} \mathcal{A}^{(r,k)} \to T\boldsymbol{Y}_\zeta \; : (\boldsymbol{y}, \bar{\Xi}) \mapsto \hat{\Xi}(\boldsymbol{y}) \,, \tag{3}$$

where, for any $\boldsymbol{y} \in \boldsymbol{Y}_\zeta$, one sets: $\hat{\Xi}(\boldsymbol{y}) = \frac{d}{dt}[(\Phi_{\zeta\, t})(\boldsymbol{y})]_{t=0}$, and $\Phi_{\zeta\, t}$ denotes the (local) flow corresponding to the gauge-natural lift of Φ_t.

This mapping fulfils the following properties (see [17]):

1. $\mathfrak{G}$ is linear over $id_{\boldsymbol{Y}_\zeta}$;

2. we have $T\pi_\zeta \circ \mathfrak{G} = id_{T\boldsymbol{X}} \circ \bar{\pi}^{(r,k)}$, where $\bar{\pi}^{(r,k)}$ is the natural projection $\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} \mathcal{A}^{(r,k)} \to T\boldsymbol{X}$;

3. for any pair $(\bar{\Lambda}, \bar{\Xi}) \in \mathcal{A}^{(r,k)}$, we have $\mathfrak{G}([\bar{\Lambda}, \bar{\Xi}]) = [\mathfrak{G}(\bar{\Lambda}), \mathfrak{G}(\bar{\Xi})]$.

2.2 Lie derivative

Definition 3 Let γ be a (local) section of $\boldsymbol{Y}_\zeta$, $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ and $\hat{\Xi}$ its gauge-natural lift. Following [17] we define the *generalized Lie derivative* of γ along the vector field $\hat{\Xi}$ to be the (local) section $\pounds_{\bar{\Xi}}\gamma : \boldsymbol{X} \to V\boldsymbol{Y}_\zeta$, given by $\pounds_{\bar{\Xi}}\gamma = T\gamma \circ \xi - \hat{\Xi} \circ \gamma$. □

Remark 2 The Lie derivative operator acting on sections of gauge-natural bundles satisfies the following properties:

1. for any vector field $\bar{\Xi} \in \mathcal{A}^{(r,k)}$, the mapping $\gamma \mapsto \pounds_{\bar{\Xi}}\gamma$ is a first–order quasilinear differential operator;

2. for any local section γ of $\boldsymbol{Y}_\zeta$, the mapping $\bar{\Xi} \mapsto \pounds_{\bar{\Xi}}\gamma$ is a linear differential operator;

3. we can regard $\pounds_{\bar{\Xi}} : J_1\boldsymbol{Y}_\zeta \to V\boldsymbol{Y}_\zeta$ as a morphism over the basis $\boldsymbol{X}$. By using the canonical isomorphisms $VJ_s\boldsymbol{Y}_\zeta \simeq J_sV\boldsymbol{Y}_\zeta$ for all s, we have $\pounds_{\bar{\Xi}}[j_s\gamma] = j_s[\pounds_{\bar{\Xi}}\gamma]$, for any (local) section γ of $\boldsymbol{Y}_\zeta$ and for any (local) vector field $\bar{\Xi} \in \mathcal{A}^{(r,k)}$.

□

2.3 Variational sequences

For the sake of simplifying notation, sometimes, we will omit the subscript ζ, so that all our considerations shall refer to $\boldsymbol{Y}$ as a gauge-natural bundle as defined above.

Remark 3 According to [19, 20, 29], the fibered splitting (1) yields the *sheaf splitting* $\mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^p \mathcal{C}^{p-t}_{(s+1,s)} \wedge \mathcal{H}^t_{s+1}$, which restricts to the inclusion $\Lambda^p_s \subset \bigoplus_{t=0}^p \mathcal{C}^{p-t}{}_s \wedge \mathcal{H}^{t,}{}^h_{s+1}$, where $\Lambda, \mathcal{H}, \mathcal{C}$ are respectively sheaves of differential, horizontal and contact forms in the sense of [19], $\mathcal{H}^{p,}{}^h_{s+1} := h(\Lambda^p_s)$ for $0 < p \leq n$ and the surjective map h is defined to be the restriction to Λ^p_s of the projection of the above splitting onto the non–trivial summand with the highest value of t (see *e.g.* [25, 26] for notation). □

By an abuse of notation, let us denote by $d\ker h$ the sheaf generated by the presheaf $d\ker h$ in the standard way. We set $\Theta^*_s := \ker h + d\ker h$.

In [19] it was proved that the following sequence is an exact resolution of the constant sheaf $\mathbb{R}_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$:

$$0 \longrightarrow \mathbb{R}_Y \longrightarrow \Lambda^0_s \xrightarrow{\mathcal{E}_0} \Lambda^1_s/\Theta^1_s \xrightarrow{\mathcal{E}_1} \Lambda^2_s/\Theta^2_s \xrightarrow{\mathcal{E}_2} \dots \xrightarrow{\mathcal{E}_{I-1}} \Lambda^I_s/\Theta^I_s \xrightarrow{\mathcal{E}_I} \Lambda^{I+1}_s \xrightarrow{d} 0$$

Definition 4 The above sequence, where the highest integer I depends on the dimension of the fibers of $J_s\boldsymbol{Y} \to \boldsymbol{X}$ (see, in particular, [19]), is said to be the s–th order *variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$. □

For practical purposes we shall limit ourself to consider the truncated variational sequence:

$$0 \longrightarrow I\!R_Y \longrightarrow \mathcal{V}^0_s \xrightarrow{\mathcal{E}_0} \mathcal{V}^1_s \xrightarrow{\mathcal{E}_1} \dots \xrightarrow{\mathcal{E}_n} \mathcal{V}^{n+1}_s \xrightarrow{\mathcal{E}_{n+1}} \mathcal{E}_{n+1}(\mathcal{V}^{n+1}_s) \xrightarrow{\mathcal{E}_{n+2}} 0\,,$$

where, following [29], the sheaves $\mathcal{V}^p_s := \mathcal{C}^{p-n}_s \wedge \mathcal{H}^{n,h}{}_{s+1}/h(d\ker h)$ with $0 \leq p \leq n+2$ are suitable representations of the corresponding quotient sheaves in the variational sequence by means of sheaves of sections of tensor bundles.

Let $\alpha \in \mathcal{C}^1_s \wedge \mathcal{H}^{n,h}{}_{s+1} \subset \mathcal{V}^{n+1}_{s+1}$. Then there is a unique pair of sheaf morphisms ([15, 18, 29])

$$E_\alpha \in \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \qquad F_\alpha \in \mathcal{C}^1_{(2s,s)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \tag{4}$$

such that $(\pi^{2s+1}_{s+1})^*\alpha = E_\alpha - F_\alpha$ and F_α is *locally* of the form $F_\alpha = d_H p_\alpha$, with $p_\alpha \in \mathcal{C}^1_{(2s-1,s-1)} \wedge \mathcal{H}^{n-1}{}_{2s}$.

Let $\eta \in \mathcal{C}^1_s \wedge \mathcal{C}^1_{(s,0)} \wedge \mathcal{H}^{n,h}{}_{s+1} \subset \mathcal{V}^{n+2}_{s+1}$; then there is a unique morphism

$$K_\eta \in \mathcal{C}^1_{(2s,s)} \otimes \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}{}_{2s+1}$$

such that, for all $\Xi : \boldsymbol{Y} \to V\boldsymbol{Y}$, $E_{j_s\Xi\rfloor\eta} = C^1_1(j_{2s}\Xi \otimes K_\eta)$, where C^1_1 stands for tensor contraction on the first factor and $\rfloor$ denotes inner product (see [18, 29]). Furthermore, there is a unique pair of sheaf morphisms

$$H_\eta \in \mathcal{C}^1_{(2s,s)} \wedge \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \quad G_\eta \in \mathcal{C}^2_{(2s,s)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \tag{5}$$

such that $\left(\pi^{2s+1}_{s+1}\right)^*\eta = H_\eta - G_\eta$ and $H_\eta = \frac{1}{2}\,A(K_\eta)$, where A stands for antisymmetrisation. Moreover, G_η is *locally* of the type $G_\eta = d_H q_\eta$, where $q_\eta \in \mathcal{C}^2_{(2s-1,s-1)} \wedge \mathcal{H}^{n-1}{}_{2s}$; hence $[\eta] = [H_\eta]$ [18, 29].

Remark 4 A section $\lambda \in \mathcal{V}^n_s$ is just a Lagrangian of order $(s+1)$ of the standard literature. Furthermore $\mathcal{E}_n(\lambda) \in \mathcal{V}^{n+1}_s$ coincides with the standard higher order Euler–Lagrange morphism associated with λ. □

3 Variations and generalized Jacobi morphisms

We recall some previous results concerning the representation of *generalized gauge-natural Jacobi morphisms* in variational sequences and their relation with the second variation of a generalized gauge-natural invariant Lagrangian. We consider *formal variations* of a morphism as *multiparameter deformations* and relate the second variation of the Lagrangian λ to the Lie derivative of the associated Euler–Lagrange morphism and to the generalized Bianchi morphism; see [25] for details.

Definition 5 Let $\pi : \boldsymbol{Y} \to \boldsymbol{X}$ be any bundle and let $\alpha : J_s\boldsymbol{Y} \to \overset{p}{\wedge} T^* J_s\boldsymbol{Y}$ and $L_{j_s\Xi_k}$ be the Lie derivative operator acting on differential fibered morphism. Let Ξ_k, $1 \leq k \leq i$, be (vertical) variation vector fields on $\boldsymbol{Y}$ in the sense of [8, 9, 12, 25]. We define the i–th formal variation of the morphism α to be the operator: $\delta^i\alpha = L_{j_s\Xi_1} \dots L_{j_s\Xi_i}\alpha$.

Remark 5 Let $\alpha \in (\mathcal{V}^n_s)_{\boldsymbol{Y}}$.
We have $\delta^i[\alpha] := [\delta^i \alpha] = [L_{\Xi_i} \dots L_{\Xi_1} \alpha] = \mathcal{L}_{\Xi_i} \dots \mathcal{L}_{\Xi_1}[\alpha]$. □

Definition 6 We call the operator δ^i the *i–th vertical variational derivative*. □

As a consequence of the Second Noether Theorem (see *e.g.* [10, 25]), the following characterization of the second variational vertical derivative of a generalized Lagrangian in the variational sequence holds true.

Proposition 1 *Let $\lambda \in (\mathcal{V}^n_s)_{\boldsymbol{Y}}$ and let Ξ be a variation vector field; then we have*

$$\delta^2 \lambda = [\mathcal{E}_n(j_{2s}\Xi \rfloor h\delta\lambda) + C^1_1(j_{2s}\Xi \otimes K_{hd\delta\lambda})] . \tag{6}$$

3.1 Generalized *gauge-natural* Jacobi morphisms

Let now specify $\boldsymbol{Y}$ to be a gauge-natural bundle and let $\hat{\Xi} \equiv \mathfrak{G}(\bar{\Xi})$ be a variation vector field associated to some $\bar{\Xi} \in \mathcal{A}^{r,k}$. Let us consider $j_s\hat{\Xi}_V$, *i.e.* the vertical part according to the splitting (1). We shall denote by $j_s\bar{\Xi}_V$ the induced section of the vector bundle $\mathcal{A}^{(r+s,k+s)}$. The set of all sections of this kind defines a vector subbundle of $J_s\mathcal{A}^{(r,k)}$, which by a slight abuse of notation (since we are speaking about vertical parts with respect to the splitting (1)), we shall denote by $VJ_s\mathcal{A}^{(r,k)}$.

By applying an abstract result due to Kolář, see [15], concerning a global decomposition formula for vertical morphisms, and by using Proposition 1 we can prove the following.

Lemma 1 *Let λ be a Lagrangian and $\hat{\Xi}_V$ a variation vecrtor field. Let us set $\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) := C^1_1(j_{2s}\hat{\Xi} \otimes K_{hd\mathcal{L}_{j_{2s}\bar{\Xi}_V}\lambda}) \equiv E_{j_s\hat{\Xi}\rfloor hd\mathcal{L}_{j_{2s+1}\bar{\Xi}_V}\lambda}$. Let D_H be the horizontal differential on $J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)}$. Then we have:*

$$(\pi^{4s+1}_{2s+1})^* \chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) = E_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)} + F_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)} , \tag{7}$$

where

$$E_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V} : J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X}) ,$$

and locally $F_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)} = D_H M_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)}$ with

$$M_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)} : J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s-1}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X}) .$$

Proof. As a consequence of linearity properties of both $\chi(\lambda, \Xi)$ and the Lie derivative operator $\pounds$ we have $\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) : J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X})$ and $D_H \chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$. Thus Kolář's decomposition formula can be applied. QED

Definition 7 We call the morphism $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) := E_{\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$ the *gauge-natural generalized Jacobi morphism* associated with the Lagrangian λ and the variation vector field $\bar{\Xi}$ (it depends on the gauge-natural lift $\mathfrak{G}(\bar{\Xi})_V$). Coordinate expressions can be found *e.g.* in [25]. □

The morphism $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ is a *linear* morphism with respect to the projection $J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to J_{4s}\boldsymbol{Y}_\zeta$. We have then (see [25]) the following.

Theorem 1 *Let $\delta^2_{\mathfrak{G}}\lambda$ be the variation of λ with respect to vertical parts of gauge-natural lifts of infinitesimal principal automorphisms. We have:*

$$\mathfrak{G}(\bar{\Xi})_V \rfloor \mathcal{E}_n(\mathfrak{G}(\bar{\Xi})_V \rfloor \mathcal{E}_n(\lambda)) = \delta^2_{\mathfrak{G}}\lambda = \mathcal{E}_n(\mathfrak{G}(\bar{\Xi})_V \rfloor h(d\delta\lambda)) \,. \tag{8}$$

Proposition 2 *Let $\lambda \in \mathcal{V}^n_s$ be a gauge-natural Lagrangian and $(\hat{\Xi}, \xi)$ a gauge-natural symmetry of λ. Then we have $0 = -\pounds_{\bar{\Xi}} \rfloor \mathcal{E}_n(\lambda) + d_H(-j_s \pounds_{\bar{\Xi}} \rfloor p_{d_V\lambda} + \xi \rfloor \lambda)$.*

(See *e.g.* [28] for geometric field theories and, in particular, [10, 26]).

3.2 Bianchi morphism and curvature

Generalized Bianchi identities for field theories are necessary and (locally) sufficient conditions for the Noether conserved current $\epsilon = -j_s \pounds_{\bar{\Xi}} \rfloor p_{d_V\lambda} + \xi \rfloor \lambda$ to be not only closed but also the divergence of a skew-symmetric (tensor) density along solutions of the Euler–Lagrange equations (see *e.g.* [28]).

Let now λ be a gauge-natural Lagrangian. We set

$$\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv \pounds_{\bar{\Xi}} \rfloor \mathcal{E}_n(\lambda) : J_{2s}\boldsymbol{Y}_\zeta \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X}) \,. \tag{9}$$

The morphism $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ so defined is a generalized Lagrangian associated with the field equations of the original Lagrangian λ . It has been considered in applications *e.g.* in General Relativity (see [7] and references quoted therein). We have $D_H\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$ and by the linearity of $\pounds$ we can regard $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ as the extended morphism $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) : J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)}) \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X})$. Thus we can state the following [25].

Lemma 2 *Let $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ be as above. Then we have* globally

$$(\pi^{4s+1}_{s+1})^*\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = \beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) + F_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} \,,$$

where

$$\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv E_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} : \tag{10}$$

$$J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X}) \tag{11}$$

and, locally, *$F_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} = D_H M_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$, with*

$$M_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} :$$

$$J_{4s-1}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s-1}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_{2s-1}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X}) \,.$$

Definition 8 We call the global morphism $\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) := E_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$ the *generalized Bianchi morphism* associated with the Lagrangian λ and the variation vector field $\bar{\Xi}$. □

Remark 6 For any $(\bar{\Xi}, \xi) \in \mathcal{A}^{(r,k)}$, as a consequence of the gauge-natural invariance of the Lagrangian, the morphism $\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv \mathcal{E}_n(\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V))$ is *locally* identically vanishing. We stress that these are just *local generalized Bianchi identities* [1]. □

Let $\mathfrak{K}$ be the *kernel* of $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V)$. As a consequence of the considerations above we have the following result [25].

Theorem 2 *The generalized Bianchi morphism is globally vanishing for the variation vector field $\bar{\Xi}$ if and only if $\delta^2_{\mathfrak{G}}\lambda \equiv \mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$,* i.e. *if and only if $\mathfrak{G}(\bar{\Xi})_V \in \mathfrak{K}$.*

From now on we shall write $\omega(\lambda, \mathfrak{K})$ to denote $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ when $\mathfrak{G}(\bar{\Xi})_V$ belongs to $\mathfrak{K}$. Analogously for β and other morphisms.

As we already recalled, the classical Jacobi equation for geodesics of a Riemannian manifold defines in fact the Riemann curvature tensor of the metric g; because of this it was suggested in [2] that the second variation and the generalized Jacobi equations define the 'curvature' of any given variational principle. This general concept of 'curvature' takes, for example, a particularly significant form in the case of generalized harmonic Lagrangians, giving rise to suitable 'curvature tensors' which satisfy suitable 'generalized Bianchi identities' [2]. Following this guideline, we shall provide a definition of the curvature of a given gauge-natural invariant variational principle.

First of all let us make the following important consideration.

Proposition 3 *For each $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ such that $\bar{\Xi}_V \in \mathfrak{K}$, we have*

$$\mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K}) = -D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})}) . \tag{12}$$

PROOF. The horizontal splitting gives us $\mathcal{L}_{j_s\bar{\Xi}}\omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K}) + \mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda, \mathfrak{K})$. Furthermore, $\omega(\lambda, \mathfrak{K}) \equiv -\pounds_{\bar{\Xi}}\rfloor\mathcal{E}_n(\lambda) = \mathcal{L}_{j_s\bar{\Xi}}\lambda - d_H(-j_s\pounds_{\bar{\Xi}}\rfloor p_{d_V\lambda} + \xi\rfloor\lambda)$; so that

$$\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s\bar{\Xi}_V}\mathcal{L}_{j_s\bar{\Xi}}\lambda = \mathcal{L}_{j_s[\bar{\Xi}_V, \bar{\Xi}_H]}\lambda .$$

On the other hand we have

$$\mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s[\bar{\Xi}_H, \bar{\Xi}_V]}\lambda = -\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda, \mathfrak{K}) .$$

Recall now that from the Theorem above we have $\bar{\Xi}_V \in \mathfrak{K}$ if and only if $\beta(\lambda, \mathfrak{K}) = 0$. Since

$$\begin{aligned}\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda, \mathfrak{K}) &= -\pounds_{\bar{\Xi}_V}\rfloor\mathcal{E}_n(\omega(\lambda, \mathfrak{K})) + D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})}) = \\ &= \beta(\lambda, \mathfrak{K}) + D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})}) ,\end{aligned}$$

we get the assertion. $\square$

It is easy to realize that, because of the gauge-natural invariance of the generalized Lagrangian λ, the new generalized Lagrangian $\omega(\lambda, \mathfrak{K})$ is gauge-natural invariant too, *i.e.* $\mathcal{L}_{j_s\bar{\Xi}}\,\omega(\lambda, \mathfrak{K}) = 0$.

Even more, we can state the following.

Proposition 4 *Let $\bar{\Xi}_V \in \mathfrak{K}$. We have*

$$\mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K}) = 0\,. \tag{13}$$

PROOF. In fact, when $\bar{\Xi}_V \in \mathfrak{K}$, since $\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda, \mathfrak{K}) = [\bar{\Xi}_V, \bar{\Xi}_V]\rfloor\mathcal{E}_n(\lambda) + \bar{\Xi}_V\rfloor\mathcal{L}_{j_s\bar{\Xi}_V}\mathcal{E}_n(\lambda) = 0$, we have

$$0 = \mathcal{L}_{j_s\bar{\Xi}}\omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda, \mathfrak{K}) + \mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K})\,. \tag{14}$$

$\boxed{QED}$

As a quite relevant byproduct we get also the following (this result can be compared with [3]).

Corollary 1 *Let $\bar{\Xi}_V \in \mathfrak{K}$. We have the* covariant *conservation law*

$$D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})}) = 0\,. \tag{15}$$

PROOF.

$$\begin{aligned}0 = \mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K}) = -\beta(\lambda, \mathfrak{K}) - \\ D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})}) = \\ -D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})})\,.\end{aligned}$$

$\boxed{QED}$

Definition 9 We define the covariantly conserved current

$$\mathcal{H}(\lambda, \mathfrak{K}) = -j_s\pounds\rfloor p_{D_V\omega(\lambda,\mathfrak{K})}\,, \tag{16}$$

to be a Hamiltonian form for ω on the Legendre bundle $\Pi \equiv V^*(J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)}) \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X})$ (in the sense of [21]). $\square$

Let Ω be the multisimplectic form on Π. It is well known that every Hamiltonian form $\mathcal{H}(\lambda, \mathfrak{K})$ admits a canonical Hamiltonian connection $\gamma_{\mathcal{H}}(\lambda, \mathfrak{K})$ such that $\gamma_{\mathcal{H}}(\lambda, \mathfrak{K})\rfloor\Omega = d\mathcal{H}(\lambda, \mathfrak{K})$. Let then $\gamma_{\mathcal{H}}(\lambda, \mathfrak{K})$ be the corresponding Hamiltonian connection form (see [21]).

Definition 10 We define the *curvature* of the given gauge-natural invariant variational principle to be the curvature of the Hamiltonian connection form $\gamma_{\mathcal{H}}(\lambda, \mathfrak{K})$. $\square$

We claim that the Euler–Lagrange and the generalized Jacobi equations for λ can be related with the zero curvature equations for the Hamiltonian connection form $\gamma_{\mathcal{H}}(\lambda, \mathfrak{K})$. According to [2, 8, 9], they must be, in fact, the Lagrangian counterpart of Hamilton equations for the Lagrangian ω obtained by contracting Euler–Lagrange equations with Jacobi variation vector fields. Investigations are in progress and results will appear elsewhere [13].

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